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Particle Masses, Force Constants, and Spin(8)

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From a number of qualitative conjectures, the constants m_e, c, $\hbar$, and a spin(8) gauge field theory, I derive the following particle masses (quark masses are constituent masses) and force constants: up quark mass = 312.7542 MeV; down quark mass = 312.7542 MeV; proton mass = 938.2626 MeV; neutrino masses (all types) = 0; muon mass = 104.76 MeV; strange quark mass = 523 MeV; charmed quark mass = 1989 MeV; tauon mass = 1877 MeV; bottom quark mass = 5631 MeV; top quark mass = 129.5 GeV; W^+ mass = 80.87 GeV; W^- mass = 80.87 GeV; W_0 mass = 99.04 GeV; fine structure constant $\alpha = 1/137.036082$; weak constant times the proton mass squared $fM_p^2 = 0.97 \times 10^{-5}$; color constant = 0.6286. From the pion mass in addition, I derive the Planck mass ≈ $(1\text{–}1.6) \times 10^{19}$ GeV, so that the gravitational constant times the proton mass squared $GM_p^2 \approx (3.6\text{–}8.8) \times 10^{-39}$.

With certain qualitative conjectures, spin(8) gauge field theory can be made to give values for particle masses and force constants that are close to currently accepted experimental values. I do not know how to prove the conjectures. If they can be shown to be true, then spin(8) gauge field theory should be a good candidate for a unified theory of electromagnetism, the weak force, the color force, and gravitation. If they cannot be shown to be true, then the values calculated herein should be considered to be nothing more than interesting numerology.

As is discussed in Appendix A, spin(8) gauge field theory has a natural lattice gauge theory structure. It is assumed that the gauge bosons of the four forces are carried by the links of the space-time lattice and that the fermion matter particles and antiparticles are at the vertices of the lattice.

Spin(8) has a 28-dimensional Lie algebra and has a Weyl group with 192 elements.

The Weyl group of spin(8) (192 elements) can be decomposed as the semidirect product of the Weyl groups of sp(2) (eight elements), $SU(3)$

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(six elements), spin(4) (four elements), and the maximal torus $U(1)^4$ (identity).

Sp(2) is isomorphic to spin(5) and acts naturally on S^4. Its Lie algebra is isomorphic to that of the de Sitter group and has ten infinitesimal generators. Those ten infinitesimal generators are here identified as gravitons, the carriers of the gravitational force.

$SU(3)$ acts naturally on CP^2. Its Lie algebra has eight infintesimal generators, which are here identified as gluons, the carriers of the color force.

Spin(4) is isomorphic to the direct product $SU(2)\times SU(2)$ and acts naturally on $S^2\times S^2$. Its Lie algebra has six infinitesimal generators.

Conjecture 1. By spontaneous symmetry breaking, the six infinitesimal generators of spin(4) act like the three infinitesimal generators of $SU(2)$, but acquire mass related to the masses of the first-generation fermions.

Comment. Consider two links of the space-time lattice connected by a common vertex. Consider the first link as carrying a massless gauge boson corresponding to any of the six infinitesimal generators of spin(4). Then spontaneous symmetry breaking should require that the gauge boson carried by the second link be such that the net result of the two links taken together should be one of the three infinitesimal generators of $SU(2)$. The three generators of $SU(2)$ should then correspond to the W^+, W^-, and W^0 weak bosons. Their masses should come from the fermion particles and antiparticles associated with the vertex joining the two links. Such a mechanism for spontaneous symmetry breaking would have no leftover Higgs scalar particles, and is therefore distinguishable from the standard Weinberg–Salam theory. It is more closely related to purely geometric theories (Finkelstein et al., 1963).

Pursuant to Conjecture 1, the six infinitesimal generators of spin(4) are here identified with the weak bosons W^+, W^-, and W^0.

The maximal torus $U(1)^4$ acts naturally on $T^4=(S^1)^4$. Its four infinitesimal generators are here identified with the four components (one time and three space) of the photon, the carrier of the electromagnetic force.[2]

[2]"We have so far used the term "photon" rather loosely; actually there are *four different kinds* of photons that can be exchanged between the electrons, which correspond to the four possible directions of polarization x, y, z, and t. By a suitable transformation of the representation, these can be replaced by an *instantaneously acting* Coulomb interaction and *two* kinds of photons which are of the familiar kind, polarized transverse to the direction of motion and propagated at the speed of light. It is thus found that the inverse-square law of static interaction and the delayed dynamical action between charges are both accounted for by the *single process* of the transmission of four-component "photons" between the charges" (Leighton, 1959).

Now consider the elementary spinor irreducible representations of spin(8), and denote them by s_+ and s_-. As s_+ and s_- are mirror images of each other, s_+ can be taken to the left handed and s_- to be right handed.

The Lie algebra of spin(8) can be written in terms of triples of Pauli matrices as follows (Gunaydin et al., 1973; Georgi, 1982): Let

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\otimes$ denote the direct product. Then define $o_1, \ldots, o_7$ by

$$\begin{aligned} io_1 &= -s_1 \otimes s_1 \otimes s_2, & io_4 &= s_2 \otimes s_2 \otimes s_1 \\ io_2 &= -s_1 \otimes s_2 \otimes I, & io_5 &= -s_2 \otimes s_2 \otimes s_3 \\ io_3 &= s_1 \otimes s_3 \otimes s_2, & io_6 &= s_2 \otimes I \otimes s_2 \\ & & io_7 &= -s_3 \otimes I \times I \end{aligned}$$

Form the Lie algebra of spin(8) by defining $o_{AB} = (-1/2i)[o_A, o_B]$, and noticing that the 21 independent matrices of the o_{AB} form the Lie algebra of spin(7). The 28-element Lie algebra of spin(8) is given by $o_{AB} \oplus io_A$, where A, B run from 1 to 7.

Therefore the gauge bosons corresponding to the Lie algebra infinitesimal generators can be seen as acting on spinor particles represented by triples of spinors, and an octet basis for the fermion particles upon which the left-handed representation s_+ acts can be taken to be

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdots \text{electron}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdots \text{up quarks}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdots \text{down quarks}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdots \text{neutrino}$$

This is a modification of the Harari–Shupe classification (Harari, 1979; Shuper, 1979; Adler, 1980) with the convention that the electric charge of a triple containing N of the $\binom{1}{0}$ spinors is $(-1)^N(N/3)$; that the $\binom{0}{1}$ spinor carries no electric charge; and that color charges are assigned as red if the third spinor is unlike the other two, blue if the second spinor is unlike the other two, green if the first spinor is unlike the other two, and colorless if all three spinors are alike.

Therefore the left-handed s_+ representation of spin(8) gives the leptons and quarks needed to build the first-generation particles of physics. Massless neutrinos must travel at the speed of light and cannot change their helicity, but massive quarks and electrons will move more slowly and can appear to have either helicity.

The same reasoning applied to the s_- representation gives the antileptons and antiquarks needed to build the first-generation antiparticles of physics.

Higher generations, as the muon second generation and the tauon third generation, should come from representations of the form $s_+^{k^+}$ or $s_-^{k^-}$, where k^+ and k^- are integers greater than 1.

In the lattice picture, the links carry gauge bosons and each vertex can have particles or antiparticles, either of the stable first-generation due to the elementary spinor irreducible representations s_+ and s_-, or of the unstable higher generations due to the higher-order representations.

Now consider the interactions between the gauge bosons and the particles and antiparticles. Let $A\otimes B\otimes C$ denote the particle or antiparticle. Let $'$ denote the map taking $\binom{1}{0}$ into $\binom{0}{1}$ and $\binom{0}{1}$ into $\binom{1}{0}$.

The $U(1)$ photon of electromagnetism does not carry either electric charge or color charge, and cannot change the nature of any leptons or quarks:

$$\text{photon:}\quad A\otimes B\otimes C\to A\otimes B\otimes C$$

The $SU(2)$ weak bosons are normally denoted by W^+, W^-, and W^0. However, here it is more convenient to use the convention that W' is W^+ or W^- (whichever does not change the fermion $A\otimes B\otimes C$) and W'' is W^- or W^+ (whichever does change the fermion $A\otimes B\otimes C$). The unconventional notation shows more clearly that W' and W'' correspond to Lie algebra elements of $SU(2)$ in the Weyl root space and that W^0 corresponds to the element of the Cartan subalgebra of $SU(2)$. Weak bosons can carry electric charge but not color charge. Weak bosons can change electrons into neutrinos and vice versa, and can change up quarks into down quarks and vice versa, but they cannot change leptons into quarks or quarks into leptons:

$$W'\colon\quad A\otimes B\otimes C\to A\otimes B\otimes C$$

$$W''\colon\quad A\otimes B\otimes C\to A'\otimes B'\otimes C'$$

$$W^0\colon\quad A\otimes B\otimes C\to A\otimes B\otimes C$$

W' and W'' are the elements of the Weyl group S_2 of $SU(2)$.

The $SU(3)$ gluons can carry color charge but not electric charge. Gluons can change the color charge of a quark, but they cannot change the nature of a lepton or quark:

$$\text{gluon}^1: \quad A\otimes B\otimes C \to A\otimes B\otimes C$$

$$\text{gluon}^2: \quad A\otimes B\otimes C \to A\otimes C\otimes B$$

$$\text{gluon}^3: \quad A\otimes B\otimes C \to B\otimes C\otimes A$$

$$\text{gluon}^4: \quad A\otimes B\otimes C \to B\otimes A\otimes C$$

$$\text{gluon}^5: \quad A\otimes B\otimes C \to C\otimes A\otimes B$$

$$\text{gluon}^6: \quad A\otimes B\otimes C \to C\otimes B\otimes A$$

$$\text{gluon}^7: \quad A\otimes B\otimes C \to A\otimes B\otimes C$$

$$\text{gluon}^8: \quad A\otimes B\otimes C \to A\otimes B\otimes C$$

Gluon^1 through gluon^6 are the elements of the Weyl group S_3 of $SU(3)$.[3] Gluon^7 and gluon^8 are elements of the Cartan subalgebra of $SU(3)$.

The sp(2) gravitons can carry color charge, electric charge, or both. Only gravitons can change leptons into quarks or quarks into leptons:

$$\text{graviton}^1: \quad A\otimes B\otimes C \to A\otimes B\otimes C$$

$$\text{graviton}^2: \quad A\otimes B\otimes C \to A'\otimes B\otimes C$$

$$\text{graviton}^3: \quad A\otimes B\otimes C \to A\otimes B'\otimes C$$

$$\text{graviton}^4: \quad A\otimes B\otimes C \to A\otimes B\otimes C'$$

$$\text{graviton}^5: \quad A\otimes B\otimes C \to A\otimes B'\otimes C'$$

$$\text{graviton}^6: \quad A\otimes B\otimes C \to A'\otimes B\otimes C'$$

$$\text{graviton}^7: \quad A\otimes B\otimes C \to A'\otimes B'\otimes C$$

$$\text{graviton}^8: \quad A\otimes B\otimes C \to A'\otimes B'\otimes C'$$

$$\text{graviton}^9: \quad A\otimes B\otimes C \to A\otimes B\otimes C$$

$$\text{graviton}^{10}: \quad A\otimes B\otimes C \to A\otimes B\otimes C$$

[3]Note that, as $SU(3)$ and sp(2) are rank-2 Lie groups, the root spaces of their Lie algebras are two-dimensional and there exists a 1:1 correspondence between the root vectors and the Weyl group elements that correspond to reflections in hyperplanes perpendicular to the root vectors. This useful correspondence does not exist for higher-rank Lie groups in general, and particularly does not exist for spin(8), which has rank 4 and a four-dimensional root space, and has 24 root vectors (arranged as the vertices of a 24-cell) but a Weyl group with 192 elements. Therefore it is much easier to carry out part of the analysis of this paper after decomposing spin(8) into sp(2), $SU(3)$, spin(4), and $U(1)^4$ rather than working directly with Spin(8) all the time.

Graviton^1 through graviton^8 are the elements of the Weyl group $S_2 \times Z_2^2$ of sp(2). Graviton^9 and graviton^{10} are elements of the Cartan subalgebra of sp(2).

Conjecture 2. The same type mechanism that confines the gluons that carry color charge also confines the gravitons that carry electric charge or color charge.

Comment. The colorless gluon^7 and gluon^8 of the Cartan subalgebra of $SU(3)$ may be unconfined but unobservable of everything is exactly colorless at scales that are experimentally observable. Similarly, only graviton^1 through graviton^8 need be confined as only they can carry electric charge or color charge. The neutral graviton^9 and graviton^{10} may be unconfirmed, and their interaction with mass should then give the observed long-range gravitational force.

The conjectured region of confinement of charged gravitons may be as small as the Planck length.

Partial Summary

In this paper thus far, subject to Conjecture 1 and Conjecture 2, spin(8) gauge field theory has been shown to classify the forces of electromagnetism, the weak force, the color force, and gravitation; to account for the qualitative properties of the gauge bosons; to classify the elementary fermion lepton and quark particles and antiparticles, including higher generations; to account for the qualitative pattern of electric and color charges of the fermion particles and antiparticles; and to have a natural lattice gauge theory structure. From more or less standard techniques of lattice gauge theory (Creutz, 1980), it should be possible to arrive at a general form for a Lagrangian for the theory.

Now we must calculate the particle masses and force constants that go into the Lagrangian to get specific predictions about experimental results.

Consider the space of triples of spinors that corresponds to the representation s_+, that is, the first generation particles. (A similar line of reasoning should apply to the s_- antiparticles.) Assume that the $\binom{0}{1}$ spinor has no mass. Then the neutrino, being $\binom{0}{1} \otimes \binom{0}{1} \otimes \binom{0}{1}$, is massless. What about the electron and the quarks?

The space of triples of spinors is an eight-dimensional complex space with infinite volume. If the mass of a particle is to be related to its "volume" in the space of triples of spinors, then calculation of ratios of particle masses requires the mapping of that space into a bounded domain. Such a bounded domain must also be an eight-dimensional complex space. Consider the irreducible symmetric bounded domain of type IV_8, denoted by D^8. It is isomorphic to $SO(10)/SO(8)\times SO(2)$.[4] Denote the Silov boundary of D^8 by Q^8. Q^8 is an eight-dimensional real space (Hua, 1963).

Conjecture 3. The mass of a first-generation electron or quark is proportional to two factors: the number of gravitons that are related to it and the volume of the part of Q^8 that is related to it.

Comment. The meaning of "related" is made clear in the analysis that follows. I do not know why Q^8 works, but it has the right dimension and gives results that are pretty well in accord with experiment.

Of the ten gravitons, graviton^9 and graviton^{10} are in the Cartan subalgebra for sp(2) and do not carry any color or electric charge. They are not considered to be related to either the electron or the quarks.

Graviton^2 through graviton^7 carry color charge. The six of them are therefore considered to be related to the quarks.

Graviton^1 and graviton^8 carry no color charge, but may carry electric charge. The two of them are therefore considered to be related to the electron. However, by interaction with the first-generation electron, only one of the two can produce a massive particle (the electron), while the other will produce a massless neutrino. Therefore only one graviton is related in a mass-producing way to the first-generation electron. (Note that this line of reasoning does not apply to higher-generation massive leptons, where both of the gravitons are related to the massive lepton in a mass-producing way.)

Therefore the graviton number factor ratio of a first-generation quark to the first-generation electron is 6:1.

What about the Q^8 volume factor? Consider the red down quark. The same analysis would apply to any of the first-generation quarks. By the color force, the red down quark can be taken into a blue down quark or a green down quark. By the weak force, the red down quark can be taken into the red up quark. By both the color and weak forces, the red down quark can be taken into a blue up quark or a green up quark. Although the weak and color forces cannot take a quark into an electron or neutrino, the quarks can combine to form a proton (two ups and a down) or a neutron (two downs and an up). The proton and neutron are similar to the electron and neutrino in that they are colorless spin-1/2 particles with unit electric charge or no electric charge. Therefore the red down quark (or any other first-generation quark) is taken to be related to all of Q^8, with volume $V(Q^8)$.

The electron cannot be taken into any other particle except a neutrino by the electromagnetic, weak, or color forces. As the neutrino is massless,

[4]Article 401, Symmetic Riemannian spaces, in the *Encyclopedic Dictionary of Mathematics* (MIT Press, Cambridge, Massachusetts, 1977).

the electron mass is taken to be related only to its own one-dimensional subspace of Q^8, the volume of which subspace is taken to be 1.

Therefore, if M_e = electron mass, M_u = up quark mass, and M_d = down quark mass (Hua, 1963):

$$\frac{M_u}{M_e}=\frac{M_d}{M_e}=\frac{6}{1}\frac{V(Q^8)}{1}=6V(Q^8)=2\pi^5$$

If M_e is taken to be 0.5110034 MeV (Lee, 1981), then $M_u=M_d=$ 312.75420 MeV. Throughout this paper the quark masses given are the constituent masses, so the proton mass $M_p=2M_u+M_d=938.2626$ MeV. Experimentally, $M_p=938.2796(27)$ MeV (Lee, 1981).

Second and higher generation fermion particles and antiparticles correspond to s_+^k and s_-^k irreducible representations of spin(8), where k is 2 or greater. Where the first generation is formed by triples of spinors, the kth generation is formed by triples of k-tuples of spinors, of which there are $(2^k)^3=2^{3k}=8^k$. They combine to form the eight particles of the kth generation due to s_+^k according to the following rules:

The triple of k-tuples containing only $\binom{0}{1}$ spinors corresponds to the neutrino;

The other 2^k-1 colorless triples of k-tuples correspond to the heavy lepton;

The $3(2^k-1)$ triples of k-tuples containing exactly two k-tuples with only $\binom{0}{1}$ spinors correspond to the down-type quarks, such as the strange or bottom quarks;

The remaining $2^{3k}-2^{k+2}+3$ triples of k-tuples correspond to the up-type quarks, such as the charmed or top quarks.

The kth generation antiparticles due to s_-^k are formed similarly.

To calculate the second generation fermion masses, consider the left-handed s_+^2 particles corresponding to triples of pairs of spinors.

The massless muon neutrino corresponds to $\binom{00}{11}\otimes\binom{00}{11}\otimes\binom{00}{11}$.

Conjecture 4. The masses of second-generation and higher-generation heavy leptons are given by comparing the symmetry groups of the elements of the triple of the heavy lepton with the symmetry groups of the fermions of lower generations, the mass ratio being the ratio of the sizes of the symmetry groups; the masses of the down-type quarks are given by multiplying the heavy lepton mass by the high-generation graviton number factor of $3=6/2$, plus any mass of the down-type quark of the next lower generation that does not contribute to the heavy lepton mass.

The masses of the up-type quarks are given by multiplying the mass of the down-type quark by the ratio of the number of triples of k-tuples for up-type quarks to the number of triples of k-tuples for down-type quarks, $(2^{3k}-2^{k+2}+3)/[3(2^k-1)]$, for kth generation quarks; and the neutrino, containing only spinors of the type $\binom{0}{1}$, is massless.

Comment. The high (at least second) generation graviton number factor is 3 because, unlike the first generation in which graviton^8: $A\otimes B\otimes C\to A'\otimes B'\otimes C'$ must take the electron $\binom{1}{0}\otimes\binom{1}{0}\otimes\binom{1}{0}$ into the neutrino $\binom{0}{1}\otimes\binom{0}{1}\otimes\binom{0}{1}$, graviton^8 can take a muon (heavy lepton) $\binom{10}{01}\otimes\binom{10}{01}\otimes\binom{10}{01}$ into another muon $\binom{01}{10}\otimes\binom{01}{10}\otimes\binom{01}{10}$. Therefore both graviton^1 and graviton^8 are related in a mass-producing way to the high-generation heavy lepton, and the graviton number factor ratio of a k-generation quark to a k-generation heavy lepton (k at least 2) is 6:2.

Now the calculations for high-generation heavy lepton and quark masses can be done.

The $2^2-1=3$ triples corresponding to the muon are $\binom{11}{00}\otimes\binom{11}{00}\otimes\binom{11}{00}$, $\binom{10}{01}\otimes\binom{10}{01}\otimes\binom{10}{01}$, and $\binom{01}{10}\otimes\binom{01}{10}\otimes\binom{01}{10}$. The first, made up of spinors all of the type $\binom{1}{0}$, corresponds to the electron. The other two correspond to the permutation group on two elements, S_2. S_2 has order 2, and is 1/3 the size of the color permutation group on three elements, S_3, that gives the up and down quarks their mass of 312.7542 MeV. Therefore the muon mass should be the sum of the electron mass and 1/3 of up or down quark mass, or 104.7642 MeV. The experimental value is 105.65946(24) MeV (Lee, 1981).

The strange quark mass should come from two sources. First, it should have the other 2/3 of the down quark mass that is not associated with the muon mass, or 208.5028 MeV. Second, it should have the muon mass times the high-generation graviton factor $6/2=3$, for 314.2872 MeV. The total strange quark mass should be 522.7900 MeV. The currently accepted estimated value is about 550 MeV (Isgur and Karl, 1983).

The charmed quark mass should also come from two sources. First, it should have the other 2/3 of the up quark mass that is not associated with the muon mass, or 208.5028 MeV. Second, as it corresponds to $2^6-2^4+3=51$ triples, it should have 51/9 times the muon part of the strange quark mass, or 1780.9608 MeV. The total charmed quark mass should be 1989.4636 MeV. The current estimate is about 1700 MeV (Isgur and Karl, 1983).

The right-handed s_-^2 antiparticle fermion masses are calculated in the same way.

To calculate the third generation fermion masses, consider the left-handed s_+^3 particles corresponding to triples of triples of spinors.

The massless tauon neutrino corresponds to $\binom{000}{111}\otimes\binom{000}{111}\otimes\binom{000}{111}$.

The $2^3-1=7$ triples corresponding to the tauon are colorless, so each is made up of $\binom{111}{000}$, $\binom{110}{001}$, $\binom{101}{010}$, $\binom{011}{100}$, $\binom{001}{110}$, $\binom{010}{101}$, or $\binom{100}{011}$, respectively. Therefore, the seven triples corresponding to the tauon also correspond to the electron, the red, blue, and green up quarks, and the red, blue, and green down quarks, and the mass of the tauon should be the same as the sum of the masses of the first generation massive fermion particles: 1877.036 MeV. The experimental value is 1784(4) MeV (Lee, 1981).

The bottom quark should have the tauon mass times the high generation graviton factor of 3, for 5631.108 MeV. The currently accepted estimated value is about 5200 MeV (Isgur and Karl, 1983).

The top quark corresponds to $2^9 - 2^5 + 3 = 483$ triples, so it should have 483/21 times the bottom quark mass, for 129 515.48 MeV. The current lower bound is 17 900 MeV (Lee, 1981).

The right-handed s_-^3 antiparticle fermion masses are calculated in the same way.

Similar calculations could be made for higher generations than the third. It should be noted that some down-type quarks of higher generations may have masses less than the top quark mass, and that the heavy leptons of generation $3k$ should have mass equal to the sum of the masses of the heavy lepton and quarks of the k-generation. As an easy example, calculate the masses of the 6-generation:

$$\text{Heavy lepton mass} = \text{muon mass} + 3(\text{charmed quark mass}) + 3(\text{strange quark mass}) = 7640\ \text{MeV};$$

$$\text{Down-type quark mass} = 3(\text{heavy lepton mass}) = 22\,920\ \text{MeV};$$

$$\text{Up-type quark mass} = (\text{down-type quark mass})\ (2^{18} - 2^8 + 3)/(3(2^6 - 1)) = 31\,765\,811\ \text{MeV};$$

$$\text{Neutrino mass} = 0.$$

Although the 6-generation up-type quark, at 32 TeV, may not be observed soon, it is worth noting that the 23 GeV down-type quark of 6-generation should be observed if generations higher than the third exist.

Calculation of force constants and weak boson masses requires a measure of the relative strengths of the four forces.

Conjecture 5. The relative strengths of the four forces are given by the ratios of the following volumes (Hua 1963), as well as some particle mass ratios:

Electromagnetism: $VE = V(S^1) = 2\pi$ (one VE for each of the four photon polarizations);

Weak force: $VW = V(S^2)\,V(Q^3)/[V(D^3)^{1/2}] = 2^5\pi^2(6/\pi)^{1/2}$ [one VW for each of the two $SU(2)$s in spin(4)];

Color force: $VC = V(CP^2)\,V(Q^{1,3})/[V(D^{1,3})^{1/4}] = 32\pi^4(6\pi)^{1/4}/3$;

Gravitational force: $VG = V(S^4)\,V(Q^5)/[V(D^5)^{1/4}] = 2^8\pi^4(15/2\pi)^{1/4}/9$

Comment. As S^1 corresponds to $U(1)$, the formula for VE is natural. The factor $V(S^2)$ is natural for VW, and the factor $V(CP^2)$ is natural for *VC*. However, I do not know why the factors $V(Q^3)$ and $[V(D^3)^{1/2}]$ work for *VW*. Neither do I know why the factors $V(Q^{1,3})$ and $[V(D^{1,3})^{1/4}]$, which are the volume of the Silov boundary of a domain of type $I_{1,3}$ and the fourth root of the volume of the domain of type $I_{1,3}$, are needed to calculate *VC*. I chose the factors because a similar choice seemed to work for *VG*. $V(S^4)$ is a natural factor for *VG*, but the ratio $V(Q^5)/[V(D^5)^{1/4}]$ is something that I do not fully understand. I do not know why the ratio works, but it gives answers that are close to experimental values. I would not have thought of using such a ratio, but it had been used earlier by Wyler (1971), who, as far as I know, did not know where the ratio came from either (Gilmore, 1972). Particle mass ratios will only come into play when dealing with the massive weak bosons or with the Planck mass of gravitation.

Now the rest of the calculations can be done.

As in the comment to Conjecture 1, consider the weak boson masses to come from a spontaneous symmetry breaking mechanism that uses two links connected by a vertex, with the masses of the weak bosons coming from the masses of the fermion particles and antiparticles at the vertex. Only stable first-generation fermions should be considered. The sum of the masses of the first-generation particles and antiparticles M_{F1} has been calculated to be 3.754 GeV. The sum of the masses of the weak bosons W^+, W^-, and W^0, denoted by M_W, should be M_{F1} times a ratio of the weak force strength to the electromagnetic force strength:

$$\frac{M_W}{M_{F1}} = 2\frac{VW}{2VE} = 16(6\pi)^{1/2}, \qquad \text{so that } M_W = 260.774 \text{ GeV}$$

VE is multiplied by 2 because there are two $U(1)$s for each $SU(2)$. The whole ratio is multiplied by 2 because there are two $SU(2)$s in spin(4).

To determine the masses of W^+, W^-, and W_0 individually, consider that $SU(2)$ is like S^3; S^3 has the Hopf fibration $S^1 \to S^3 \to S^2$; and S^2 should correspond to W^+ and W^-, while S^1 should correspond to W_0.

The unit sphere S^3 in R^4 contains the point (1/2, (1/2, 1/2, 1/2); the corresponding point in S^2 is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$; and the corresponding point in S^1 is $(1/\sqrt{2}, 1/\sqrt{2})$.

Let $M_{W_\pm}$ be the sum of the masses of W^+ and W^-, which masses should be equal: $M_{W_+} = M_{W_-}$.

Let M_{W_0} be the mass of the W_0.

$$\frac{M_{W_\pm}}{M_{W_0}} = \frac{V(S^2)(2/\sqrt{3})}{V(S^1)(2/\sqrt{2})} = \frac{4\pi\sqrt{2}}{2\pi\sqrt{3}} = 1.632993$$

As

$$M_W = 260.774\ \text{GeV:}\quad M_{W_\pm} = 161.73\ \text{GeV},\quad M_{W_+} = M_{W_-} = 80.87\ \text{GeV},$$

$$M_{W_0} = 99.04\ \text{GeV}$$

Experimentally, $M_{W_+} = M_{W_-} = 81(5)$ GeV (Lubkin, 1983), and $M_{W_0} = 93(4)$ GeV.[5]

The fine structure constant α can be defined by separating two electrons by their Compton wavelength, measuring their electrostatic energy of repulsion, and dividing that by the rest mass energy of an electron.[6] I calculate α by

$$\alpha = \frac{4VE}{VG} = 1/137.036082$$

VE is multiplied by 4 because there are four polarizations of the photon. Experimentally, $\alpha = 1/137.03604(11)$.[7]

The weak force constant G_W is given by

$$G_W = \frac{2VW}{VG}\,\frac{M_e^2}{(M_{W_+}^2 + M_{W_-}^2 + M_{W_0}^2)} = 2.886\times 10^{-12}$$

VW corresponds to the weak force that acts on S^2, and it is multiplied by 2 because there are two $SU(2)$s in spin(4). The VG corresponds to the sp(2) de Sitter gravitational force that acts on S^4. The ratio of squares of masses reflects the fact that the weak force is carried by the massive weak bosons. $G_W M_p^2 = 0.97\times 10^{-5}$, where M_p is the proton mass (Rosenfeld and Wightman, 1974). Experimentally, $G_W M_p^2 = 1.02\times 10^{-5}$. Note that G_W is related to the Fermi constant G_F by $G_W = G_F(M_e^2 c/\hbar^3)$.

The color force constant G_C is given by

$$G_C = \frac{VC}{VG} = \frac{3}{8}\left(\frac{4\pi^2}{5}\right)^{1/4} = 0.6286$$

The value is of the order of unity, and is in the range that is currently accepted in quantum chromodynamics (Lee, 1981).

The constant G_G for gravitation is given by

$$G_G = \frac{VG}{VG}\,\frac{M_e^2}{M_{PL}^2} = \frac{M_e^2}{M_{PL}^2}$$

[5]Rough average of values in G. Lubkin, *Physics Today* (November 1983), p. 17.
[6]See Manin (1981), p. 48.
[7]*Handbook of Chemistry and Physics, 59th edition* CRC Press, (1978-1979), pp. F-250, F-252.

Here, G_G is related to the Newton constant G_N by $G_G = G_N(M_e^2/\hbar c)$, and $M_{\rm Pl}$ denotes the Planck mass. $M_{\rm Pl}$ must be estimated in order to calculate G_G. To get a rough estimate, note that spin(8) gauge field theory has a natural lattice gauge theory structure, and estimate the mass of a one-vertex universe in spin(8) gauge field theory.

Consider a sum over all possible combinations of particle–antiparticle pairs of the first generation fermions at the one vertex. As a one-vertex lattice has no links, there are no gauge bosons to carry away any of the pairs. There are eight fermion particles and eight fermion antiparticles, for a total of 64 particle–antiparticle pairs. There are then 2^{64} combinations of particle–antiparticle pairs. A typical combination should have several quarks, several antiquarks, a few colorless quark–antiquark pairs that would be equivalent to pions, and some leptons and antileptons.

As the masses of leptons are small, ignore their contribution to the sum.

As the independent quarks and antiquarks are fermions, each could be present on the vertex only twice owing to the Pauli exclusion principle, so the total contribution to the mass of independent quarks and antiquarks (of which there are 12, each having mass of about 0.3 GeV) is only about 7.2 GeV.

Pions, colorless quark–antiquark pairs, are bosons and are not subject to the exclusion principle. Of the 64 particle–antiparticle pairs, 12 are pions, each having mass of about 0.14 GeV (Lee, 1981). A typical combination should have about six pions. If all the pions are independent, the typical combination should have mass of 0.14×6 GeV $= 0.84$ GeV. However, just as the pion mass of 0.14 GeV is less than the sum of the masses of a quark and an antiquark, $0.3 + 0.3 = 0.6$ GeV, pairs of oppositely charged pions may form a bound state of less mass than the sum of two pions masses, $0.14 + 0.14 = 0.28$ GeV. If such a bound state of negative and positive pions has a mass as small as 0.1 GeV, and if the typical combination has one such pair and four other pions, then the typical combination should have a mass of $0.14 \times 4 + 0.1 = 0.66$ GeV. Therefore the typical combination should have a mass in the range of (0.66–0.84) GeV. Summing over all 2^{64} combinations, the total mass of a one-vertex universe should be roughly in the range of $(1.217\text{–}1.550) \times 10^{19}$ GeV. The currently accepted value of the Planck mass is 1.221×10^{19} GeV (Manin, 1981; see also *Handbook of Chemistry and Physics, 59 Edition* (CRC Press, 1978–1979, pp. F-250, F-252), which is close to the estimate taking account of bound states of oppositely charged pions.

Using $M_{\rm Pl} = 1.2 \times 10^{19}$ GeV, $G_G = 1.8 \times 10^{-45}$, and $G_G M_p^2 = 6 \times 10^{-39}$, roughly.

Summary

Subject to the stated Conjectures, spin(8) gauge field theory is not only a good classification scheme but can also, from the input of the speed of light c, Planck's constant $\hbar$, and the electron mass M_e, give a fairly accurate set of particle masses and force constants.

As can be seen from the comments following the conjectures, as to several points I have nothing more than an intuitive guess as to how to go about working on a solution to the many outstanding problems.

Therefore it is fair to ask whether the tentative results of spin(8) gauge field theory are promising enough to warrant using it and working further on it. I believe they are.

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This paper is dedicated to my parents.

APPENDIX A: LATTICE GAUGE STRUCTURE

Spin(8) acts naturally on S^7. S^7 corresponds to the unit octonions. By applying the techniques of Barnsley, Geronimo, and Harrington (Barnsley et al., 1983) to octonions, a space-time lattice M_n can be constructed whose elements are vertices associated with nth-order Borel sets. M_n is constructed from S^7 by an octonionic iterated quadratic chaotic map $X_n : S^7 \to M_n$. M_n is a four-dimensional lattice, rather than a manifold, so that spin(8) gauge field theory naturally has a base lattice, rather than a base manifold, and spin(8) gauge field theory has a natural lattice gauge structure.

Construct a principal fibre bundle $[P, p, M_n, \text{spin}(8)]$ using X_n and considering "dense in" to be equivalent to "equals to" for the purpose of physics. To do so, define the projection p locally as the composition of the maps:

$M_n \times \text{spin}(8) \to M_n \times [\text{spin}(8)/\text{spin}(7)]$	by quotient map
$M_n \times [\text{spin}(8)/\text{spin}(7)] \to M_n \times S^7$	by defining map of S^7
$M_n \times S^7 \to M_n \times M_n$	by X_n
$M_n \times M_n \to M_n$	by diagonal map

For any lattice vertex s in M_n, $p^{-1}(s)$ is dense in $(s, \mathrm{spin}(8))$. Let P be locally $M_n \times \mathrm{spin}(8)$. Then $(P, p, M_n, \mathrm{spin}(8))$ is a principal fibre bundle that gives a totally unified gauge field theory of all four forces of nature. M_n corresponds to the four-dimensional space-time base lattice.

M_n has a natural quaternionic structure that gives one time dimension and three space dimensions.

To construct M_n and X_n begin by considering the map T_l: $1\mathbf{O} \to \mathbf{O}$, where $\mathbf{O}$ is the octonions with basis $\{o_0 = 1, o_1, \ldots, o_7\}$, and $T_l(o) = (o - l)^2$, where o is in $\mathbf{O}$ and l is in $[0, 2]$.

Define K_l as the set of points in $\mathbf{O}$ that do not go to infinity under the map T_l^n (T_l iterated n times) as n goes to infinity.

Define B_1 as the boundary of K_1. B_1 is the Julia set for T_1. The Julia sets for the complex plane are just the same as the intersections of octonionic Julia sets with any two-dimensional plane in octonionic 8-space that includes the octonionic real axis.

For $l = 2$, the critical value, B_2 is the closed interval $[0, 4]$ on the real axis.

For $l = 0$, B_0 is the unit sphere S^7.

Define ${}_iB_l$, where $i = 1, \ldots, 7$, as the intersection of B_l with the subspace of $\mathbf{O}$ spanned by $\{1, o_i\}$. For all l in $(0, 2]$, ${}_iB_l$ can be represented as the set of all points in $\mathbf{O}$ of the form $l + e_1(1 + e_2(l + e_3(l + \cdots)^{1/2})^{1/2})^{1/2}$, where $e_k = \pm 1$, and where $(\ )^{1/2}$ is defined by using $(-1)^{1/2} = o_i$.

Define a map ${}_iY_{l_1,l_2}$ *from* ${}_iB_{l_1}$ to ${}_iB_{l_2}$ by

$$ {}_iY_{l_1,l_2}(l_1 + e_1(l_1 + (e_2(l_1 + \cdots)^{1/2})^{1/2})^{1/2}) = l_2 + e_1(l_2 + e_2(l_2 + \cdots)^{1/2})^{1/2}. $$

${}_iY_{l_1,l_2}$ is defined for all l_1 and l_2 in $(0, 2]$ and can be extended to $l = 0$ by continuity.

Consider the case $l = 2$. ${}_iB_2 = [0, 4]$. Define a map ${}_is_n(x)$ from $[0, 4]$ to $[0, 4]$ by

$$ {}_is_n(x) = l + e_1(l + e_2(l + \cdots + e_{n-1}(l + e_n x^{1/2})^{1/2})^{1/2})^{1/2} \qquad (l = 2) $$

Let ${}_i\mathcal{S}_n$ be the set of all functions ${}_is_n$, with $i\mathcal{S}_0 = {}_is_0(x) = x$. Define $iR_n = \{{}_is_n(l) = {}_is_n(2) | {}_is_n$ is in $i\mathcal{S}_n\}$. ${}_iR_n$ has 2^n distinct points, which are just the zeros of the Chebyshev polynomial of degree 2^n, ${}_iP_{2^n}(z)$, where z is in $[0, 4]$. Denote the 2^n points of ${}_iR_n$ by ${}_iz_1 < {}_iz_2 < \cdots < {}_iz_{2^n}$, and let ${}_iz_0 = 0$. The intervals $({}_iz_{j-1}, {}_iz_j]$ form nth-order Borel sets for ${}_iB_2$. The corresponding Borel measure ${}_idm_n$ is the singular measure concentrated at the zeros of the Chebyshev polynomial ${}_iP_{2^n}(z)$ taking the value 2^{-n} at each zero.

Note particularly that T_2^n maps each nth-order Borel set densely onto the whole set ${}_iB_2 = [0, 4]$. In fact, T_2 acts as a Bernoulli shift operator for the Chebyshev measure system on ${}_iB_2$, and, as n goes to infinity, the Chebyshev measure goes to the measure defined by the distribution taking the values 0 for $x=0$, 1 for $x=4$, and $\int_0^x dy/[y(4-y)]^{1/2}$ for $0<x<4$. It is equivalent as a Bernoulli system to Lebesgue measure and the usual Borel sets on the unit interval.

Now assume that there exists a unique map Y_{l_1,l_2}: $B_{l_1} \to B_{l_2}$ for l_1, l_2 in $[0, 2]$ such that Y_{l_1,l_2} restricted to the subspace of $\mathbf{O}$ spanned by $\{1, o_i\}$ is equal to ${}_iY_{l_1,l_2}$ for all $i=1,\ldots,7$.

Define Z_n as the composition of $Y_{0,2}$ and $(T_2^n)^{-1}$. Z_n is a map from $B_0=S^7$ to B_2. B_2 is the interval $[0, 4]$ on the real axis, but B_2 has seven Chebyshev measure structures ${}_idm_n$, one for each imaginary octonion basis vector o_i.

If the o_i were not related by the octonionic multiplication, B_2 would be considered to be $[0, 4]^7$, with each factor $[0, 4]$ corresponding to one of the ${}_idm_n$. However, they are related. Pick one of the o_i. By symmetry, it can be taken to be o_1. Of the remaining 6, note that the subset $\{o_3, o_5, o_7\}$ is just, when multiplied by o_1, the subset $\{o_2, o_6, o_4\}$. Therefore, there are only four independent measure structures: ${}_1dm_n$, ${}_2dm_n$, ${}_6dm_n$, and ${}_4dm_n$, corresponding to $\{o_1, o_2, o_6, o_4\}$. Note that $\{o_1, o_2, o_6, o_4\}$ are four imaginary octonion basis vectors that are isomorphic to the quaternions, with o_1 being the "time" dimension and $\{o_2, o_6, o_4\}$ being the three "space" dimensions.

Therefore B_2 should be considered to be $[0, 4]^4$ with the $[0, 4]$ factor corresponding to ${}_1dm_n$ being considered as time and the three $[0, 4]$ factors corresponding to ${}_2dm_n$, ${}_6dm_n$, and ${}_4dm_n$ being considered as the three space dimensions, and with B_2 having a natural quaternionic structure.

The nth-order Borel sets and Chebyshev measure on the ${}_iB_2$ induce nth-order Borel sets and Chebyshev measure on B_2. T_2 then acts as a Bernoulli shift on B_2, and T_2^n maps each nth-order borel set densely onto the whole set $B_2=[0, 4]^4$. Therfore, for each Borel set s in B_2, $Z_n^{-1}(s)$ is dense in S^7.

The map Z_n is not quite the map needed, because it is a map from S^7 to B_2 and B_2 has points, not nth-order Borel sets, as its elements, so that the inverse images of elements of B_2 under Z_n are not dense in S^7 as needed to construct the spin(8) gauge field theory.

Define M_n as the lattice constructed from B_2 by identifying the nth-order Borel sets of B_2 (each with its Chebyshev measure) as the vertices of the lattice. Then define X_n: $S^7 \to M_n$ by composing Z_n with the defining map from B_2 to M_n. The nth-order Chebyshev measure induced on M_n is denoted by dm_{X_n}. It can be termed a chaotic measure since it is constructed from the chaotic process arising from the iterated maps T_l^n.

The base lattice M_n plays the role of space-time. As M_n is constructed by choosing a Borel set covering of a specified fineness corresponding to the choice of n for the Chebyshev polynomials P_{2^n}, the size of the nth-order Borel sets provides a natural ultraviolet cutoff and lattice structure for M_n.

As the nth-order Chebyshev measure for the polynomials P_{2^n} is a singular measure concentrated at the zeroes of those Chebyshev polynomials, the lattice structure of M_n has a natural singular measure that converges as n does to infinity (or as lattice spacing goes to zero) to the Chebyshev measure that is equivalent, as a Bernoulli scheme, to Lebesgue measure.

APPENDIX B: ADDITIONAL COMMENTS ON CONJECTURE 3

Consider the relationship between C^8, the space of triples of spinors, and the symmetric bounded domain D^8, isomorphic to $SO(10)/SO(8) \times SO(2)$. C^8 has eight complex dimensions but is unbounded. D^8 has eight complex dimensions and is a natural eight-dimensional generalization of the unit disk.

Identify the origin of C^8 with the identity coset of D^8. Consider the vector space of infinitesimal displacements at the origin of C^8, denoted by V_C. A basis for V_C may be identified with the eight basis triples of spinors, which have themselves been identified with the eight first-generation fermion particles: the electron; the red, blue, and green up quarks; the red, blue, and green down quarks; and the neutrino.

Consider the vector space of infinitesimal displacements at the identity coset of D^8, denoted by V_D. V_D has eight complex dimensions and describes the directions a geodesic through the origin may have (Gilmore, 1974).

Identify V_C with V_D. The stability subgroup $SO(8)$ naturally maps V_D onto itself (Gilmore, 1974). Therefore, there is a natural action of $SO(8)$ on the eight fermion particles, and the action arises naturally from the structure of D^8.

Perhaps that action of $SO(8)$ on the eight fermion particles could be identified with the action of the infinitesimal generators of the s_+ representation of spin(8) on the triples of spinors corresponding to the eight fermion particles of the first generation.

The action of the stability subgroup $SO(2)$ may be explained as being required by the complex structure of the domain D^8.

The Silov boundary of D^8, denoted by Q^8, is the set of vectors of the form $e^{it}x$, where $0 \leq t \leq \pi$ and x is a vector on the unit sphere in R^8. Q^8 has eight real dimensions and its volume $V(Q^8)$ is $\pi^5/3$. If z is in D^8 and u is in Q^8, then every continuous function $f(u)$ on Q^8 defines a harmonic function $f(z)$ on D^8 by the Poisson kernel $P(z, u)$: $f(z) = \int_{Q^8} P(z, u) f(u)\, du$. The harmonic functions on D^8 are defined by the Laplace operator of the Bergman kernel for D^8. Note that, if $'$ denotes transpose, $P(z, u) = (1/V(Q^8))(1 + |zz'|^2 - 2\bar{z}z')^4 / |(z-u)(z-u)'|^8$ (Hua, 1963).

Now define another kernel $R(z, u)$ by $R(z, u) = (1 + |zz'|^2 - 2\bar{z}z')^4 / |(z-u)(z-u)'|^8$. Assume that gravitational interaction with the particles in

Q^8 determines a continuous function $f(u)$ in Q^8. Then the mass of a particle in D^8 should be given by

$$\text{mass}(z) = k \int_{Q^8} R(z, u) f(u)\, du = kV(Q^8) f(z)$$

where k is a constant of proportionality involving the volume of the part of Q^8 that is related to the particle and the number of gravitons related to the particle.

APPENDIX C: ADDITIONAL COMMENTS ON CONJECTURE 5[8]

$S^2 = SO(3)/SO(2) = SU(2)/SO(2)$ is acted upon naturally by the weak force group $SU(2)$, so that $V(S^2)$ is a natural factor for VW. $SO(5)/SO(3) \times SO(2) = SO(5)/SU(2) \times SO(2)$ has $SU(2)$ as a stability subgroup, so it is natural that the volume $V(Q^3)$ of the Silov boundary of D^3 is a factor of VW. Q^3 has three real dimensions, D^3 has three complex dimensions, and S^2 has two real dimensions. The square root of $V(D^3)$ might be a "normalization length" relating Q^3 and S^2.

$CP^2 = SU(3)/S(U(2) \times U(1))$ is acted upon naturally by the color force group $SU(3)$, so that $V(CP^2)$ is a natural factor for VC. $SU(4)/S(U(3) \times U(1))$ has $SU(3)$ as a stability subgroup, so it is natural that the volume $V(Q^{1,3})$ of the Silov boundary of $D^{1,3}$ is a factor of VC. $Q^{1,3}$ has five real dimensions, $D^{1,3}$ has three complex dimensions, and CP^2 has four real dimensions (or two complex dimensions). The fourth root of $V(D^{1,3})$ might be a normalization length relating $Q^{1,3}$ and CP^2.

$S^4 = \mathrm{sp}(2)/\mathrm{sp}(1) \times \mathrm{sp}(1) = SO(5)/SO(4)$ is acted upon naturally by the de Sitter gravitational force group sp(2), so that $V(S^4)$ is a natural factor for VG. $SO(7)/SO(5) \times SO(2) = SO(7)/\mathrm{sp}(2) \times SO(2)$ has sp(2) as a stability subgroup, so it is natural that the volume $V(Q^5)$ of the Silov boundary of D^5 is a factor of VG. Q^5 has five real dimensions, D^5 has five complex dimensions, and S^4 has four real dimensions (or one quaternionic dimension). The fourth root of $V(D^5)$ might be a normalization length relating Q^5 and S^4.

Consider a factor space G/H. The stability subgroup H can be considered as the gauge group of a Yang–Mills theory with local gauge invariance under H (Gursey and Tze, 1980). Perhaps the volumes $V(Q^3)$, $V(Q^{1,3})$, and $V(Q^5)$ used for the weak, color, and gravitational forces are useful because those forces are related to Yang–Mills theories with local gauge invariance groups of $SU(2)$, $SU(3)$, and sp(2), respectively.

[8]See Article 401, Symmetric Riemannian Spaces, in the *Encylopedic Dictionary of Mathematics* (MIT Press, Cambridge, Massachusetts, 1977); also Hua (1963) and Gilmore (1974).

The volumes of S^2, CP^2, and S^4 would appear as volumes of natural base manifolds for Yang-Mills gauge field theories of the weak force, the color force, and gravitation.

APPENDIX D: OCTONIONS (Gunaydin and Gursey, 1973).

If the octonions **O** have basis $\{o_0 = 1, o_1, o_2, o_3, o_4, o_5, o_6, o_7\}$, the octonion multiplication table can be given as follows:

	o_1	o_2	o_3	o_4	o_5	o_6	o_7
o_1	-1	o_3	$-o_2$	o_7	$-o_6$	o_5	$-o_4$
o_2	$-o_3$	-1	o_1	o_6	o_7	$-o_4$	$-o_5$
o_3	o_2	$-o_1$	-1	$-o_5$	o_4	o_7	$-o_6$
o_4	$-o_7$	$-o_6$	o_5	-1	$-o_3$	o_2	o_1
o_5	o_6	$-o_7$	$-o_4$	o_3	-1	$-o_1$	o_2
o_6	$-o_5$	o_4	$-o_7$	$-o_2$	o_1	-1	o_3
o_7	o_4	o_5	o_6	$-o_1$	$-o_2$	$-o_3$	-1

Although the octonions are neither commutative nor associative, they satisfy the alternativity law:

Define $[x, y, z] = (xy)z - x(yz)$ for octionions x, y, z: $[x, y, z] = [z, x, y] = [y, z, x] = -[y, x, z] = -[x, z, y] = -[z, y, x]$.

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